NOTES ON

DESIGN OF

DIRECTIONAL ANTENNAS

BY

AL P. BARSIS

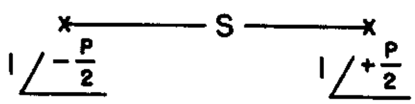
DECEMBER, 1951

NOTES ON THE DESIGN OF DIRECTIONAL ANTENNAS

1. Three-in-line

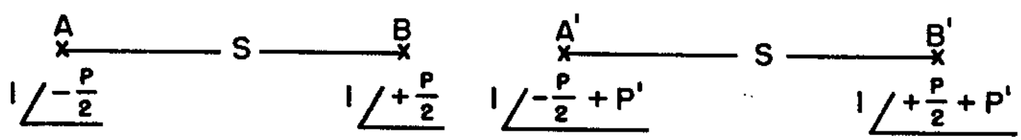
Any three (or multi) element directive array where all elements are along a straight line and equally spaced may be resolved in a series of two-element patterns which are "multiplied" together. Each two-element array may be considered as a vector of radiated field which may be combined with similar vectors representing various array combinations.

Take a two-element array with the elements spaced S electrical degrees, a phase difference of P degrees between elements, and identical fields radiated by each element which may be considered unity:



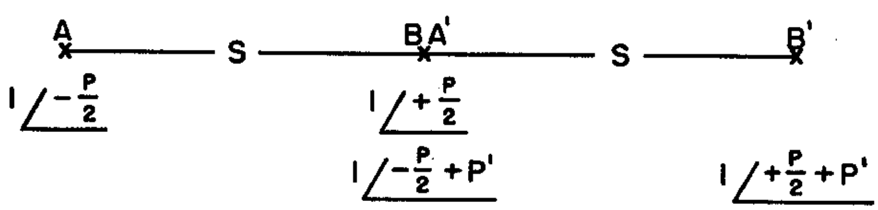
(1)

The above array may be computed by the "half-angle" formula $\cos\left(\frac{S}{2}\cos\theta + \frac{P}{2}\right)$ where the constant is omitted, and the angle θ is measured from the "leading" side of the array axis. Now picture an identical array oriented along the same axis, with the same phase difference P between its own elements, as exists between the elements of the first array, but with a phase difference P' between corresponding elements of both arrays:



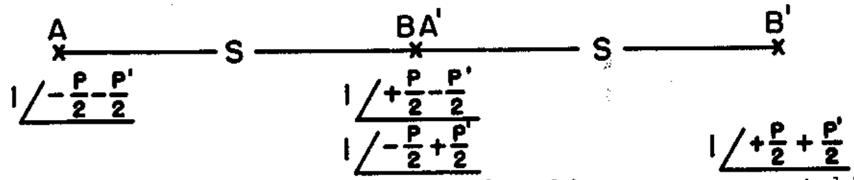
(2)

Now move the two arrays along the common axis so that elements B and A' coincide:



(3)

Subtract $\frac{P'}{2}$ from the phase of each element in order to arrive at a symmetrical (about the center) configuration:



(4)

The vector equation for this array now, taking the center element as a reference point, and zero degree azimuth along the axis in the direction of the leading tower is obviously:

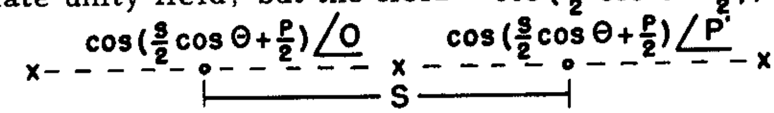
$$1\angle \frac{P}{2}-\frac{P'}{2} + 1\angle -\frac{P}{2}+\frac{P'}{2} + 1\angle S\cos\Theta+\frac{P}{2}+\frac{P'}{2} + 1\angle S\cos(\Theta-180)-(\frac{P}{2}+\frac{P'}{2})$$

which may be expanded and simplified to give

$$2\cos(\frac{P}{2}-\frac{P'}{2}) + 2\cos(S\cos\Theta + \frac{P+P'}{2})$$

An equation for the same configuration may also be obtained by the following reasoning:

In Fig. (2) above represent each array by its half-angle formula $\cos(\frac{S}{2}\cos\Theta+\frac{P}{2})$. The combination of the two similar arrays (except for the phase difference P') pictured in Fig. (4) may then be represented as a <u>two-element</u> array where each element does not radiate unity field, but the field $\cos(\frac{S}{2}\cos\Theta+\frac{P}{2})$,



(5)

and each element is assumed to be located at the midpoint of the original arrays so that the spacing between them is again S. Now the half-angle formula is again applicable to the assumed new element (each representing the original two-element array) so that the resulting three-element array is expressed by

$$\cos\left(\frac{S}{2}\cos\Theta + \frac{P}{2}\right)\cos\left(\frac{S}{2}\cos\Theta + \frac{P'}{2}\right)$$

It can be shown that

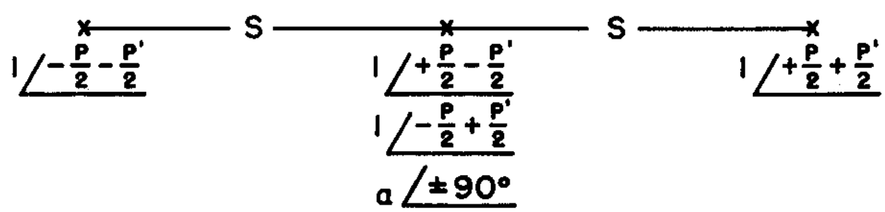
$$E = K\cos\left(\frac{S}{2}\cos\Theta + \frac{P}{2}\right)\cos\left(\frac{S}{2}\cos\Theta + \frac{P'}{2}\right)$$

is identical to

$$E = K'\left[2\cos\left(\frac{P}{2} - \frac{P'}{2}\right) + 2\cos\left(S\cos\Theta + \frac{P+P'}{2}\right)\right]$$

where K = 4 K'

The patterns discussed so far are based on unity field ratio in each component two-element array which will produce "zero" nulls. In order to make the nulls different from zero, several methods may be employed. If it is desired to have equal minima in the pattern nulls, a quadrature component may be added to the field radiated by the center tower. The component a added to (4) results in



(6)

Development of the vector equation results in

$$E = K' \sqrt{\left[2\cos\left(\frac{P}{2}-\frac{P'}{2}\right) + 2\cos\left(S\cos\Theta+\frac{P+P'}{2}\right)\right]^2 + [a]^2}$$

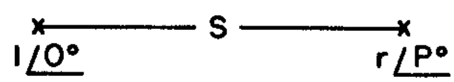
Due to the trigonometric relations the multiplication formula developed from (5) by addition of the quadrature component to the center tower will assume the form

$$E = K \sqrt{\cos^2\left(\frac{S}{2}\cos\Theta+\frac{P}{2}\right)\cos^2\left(\frac{S}{2}\cos\Theta+\frac{P'}{2}\right) + \left(\frac{a}{4}\right)^2}$$

where, again, K = 4K'

If nulls of different magnitude are desired, the development will proceed from the multiplication formula employing field ratios different from unity.

We start again with a two-element array



(7)

which gives a component equal to $\sqrt{\frac{1+r^2}{2r} + \cos(S\cos\Theta + P)}$

In order to simplify the formulas, let $\frac{1+r^2}{2r} = R$

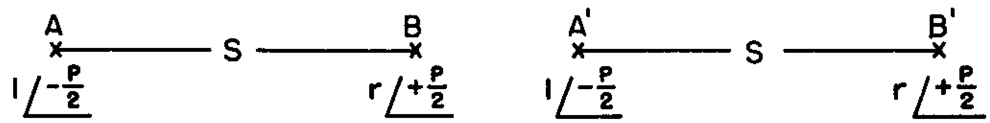
A development similar to the one discussed above for unity field components will consider a similar array where corresponding elements will be displaced in phase by P' degrees, and bear a field ratio r' with respect to the original array. Proper placement of the array will result in coincidence of elements similar to (2) and (3), and the equation

$$E = K \sqrt{\left[R + \cos(S \cos\Theta + P)\right]\left[R' + \cos(S \cos\Theta + P')\right]}$$

results.

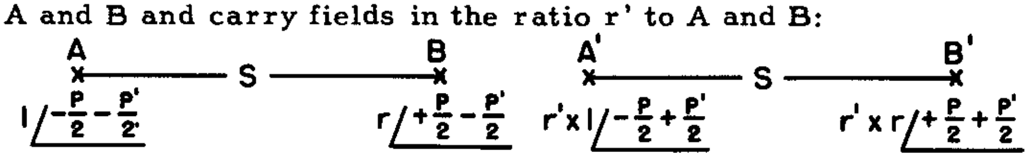
Development of the vector equation follows:

Two two-element arrays (7) are arranged collinear:



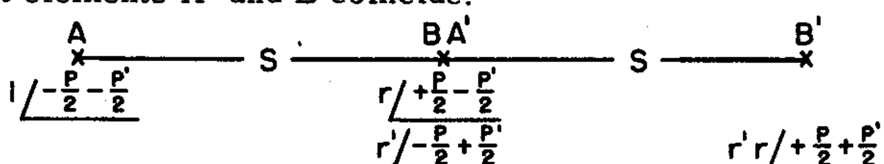
(8)

Elements A' and B' are displaced in phase P' degrees with regard to A and B and carry fields in the ratio r' to A and B:



(9)

Now let elements A' and B coincide:



(10)

Taking the center element as a reference point, the following equation may be developed:

$$E = K'\left(r\angle +\frac{P}{2}-\frac{P'}{2} + r'\angle -\frac{P}{2}+\frac{P'}{2} + r'r\angle S\cos\Theta + \left(\frac{P}{2}+\frac{P'}{2}\right) + 1\angle S\cos(\Theta-180) - \left(\frac{P}{2}+\frac{P'}{2}\right)\right)$$

Expansion and simplification results in:

$$E = K' \sqrt{\left[(r+r')\cos\left(\frac{P}{2}-\frac{P'}{2}\right)+(r'r+1)\cos\left(S\cos\Theta+\frac{P}{2}+\frac{P'}{2}\right)\right]^2 + \left[(r-r')\sin\left(\frac{P}{2}-\frac{P'}{2}\right)+(r'r-1)\sin\left(S\cos\Theta+\frac{P}{2}+\frac{P'}{2}\right)\right]^2}$$

It can be shown that

$$K' = \frac{K}{2\sqrt{rr'}}$$

The development of (10) has to be watched carefully for algebraic signs which will depend on the direction for which the reference axis is taken. The procedure illustrated above considers the leading phase side as the zero direction.

By employment of the ratios r and r', various combinations are possible as follows (only ratios are considered with the phase relations being the same in each case):

1	r r'	rr'	(10 a)
r	1 rr'	r'	(10 b)
r'	rr' 1	r	(10 c)
rr'	r' r	1	(10 d)

All above combinations result in the <u>same</u> horizontal plane pattern. If all elements are of equal height, the vertical sections will also be identical. However, if one or two elements are of different height, vertical sections <u>will be different</u> for each combination. The choice of the proper combination will then depend on the shape of the vertical section most desirable.

Similarly, although current and power distribution may be different for each possible combination, the resultant power gain is the same <u>if</u> the towers are of equal height.

It is conceivable, although usually not required or practical to add a quadrature component to the center element in (10) (or any of the variations), which would increase the field in all of the nulls. Derivation of the equations for this case would be somewhat simpler for actual numerical values than for the general case.

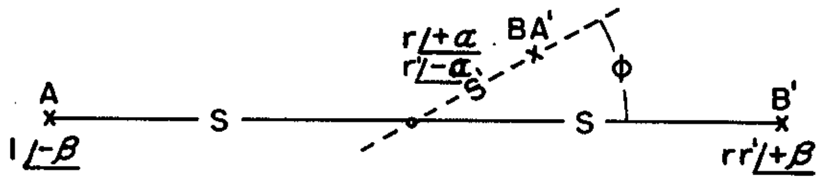
<u>2. Off-set center tower.</u>

If a slightly unsymmetrical pattern is desired, or if nulls of varying deepness are called for, it may be advisable to off-set the center-tower by a short distance, after the pattern has been developed in the manner previously described. The array (10), with its parameters simplified yields

$$\underset{1\underline{/-\beta}}{\overset{A}{\times}} \text{———} S \text{———} \underset{\substack{r\underline{/+\alpha} \\ r'\underline{/-\alpha}}}{\overset{BA'}{\times}} \text{———} S \text{———} \underset{rr'\underline{/+\beta}}{\overset{B'}{\times}} \qquad (11)$$

where $\beta = \frac{P}{2} + \frac{P'}{2}$ and $\alpha = \frac{P}{2} - \frac{P'}{2}$

Now assume the element BA' to be displaced along a line which includes an angle ϕ with the array axis by a spacing of S' electrical degrees:



(12)

For this case, only the vector sum equation can be developed. Taking the midpoint between A and B' as a reference point, zero azimuth along the axis AB', and progressing in a counter-clockwise direction (12) may be expressed as

$$E = K'\left[rr' \angle S\cos\theta + \beta + 1 \angle S\cos(\theta - 180) - \beta + r \angle S'\cos(\theta - \phi) + \alpha + r' \angle S'\cos(\theta - \phi) - \alpha \right]$$

which after expansion and simplification yields:

$$E = K' \sqrt{\left[(rr'+1)\cos(S\cos\theta + \beta) + (r+r')\cos[S'\cos(\theta - \phi) + \alpha]\right]^2 + \left[(rr'-1)\sin(S\cos\theta + \beta) + (r-r')\sin[S'\cos(\theta - \phi) + \alpha]\right]^2}$$

This formula resembles the one for the three-in-line previously developed, and permits a better estimate of the result, and is more readily adaptable to the actual computation work. Of course, field and phase values for each element may be changed arbitrarily to suit the design purpose, but equal and opposite phases in the elements A and B' have to be maintained in order to keep the formula relatively simple.

3. General 3-Element Configurations

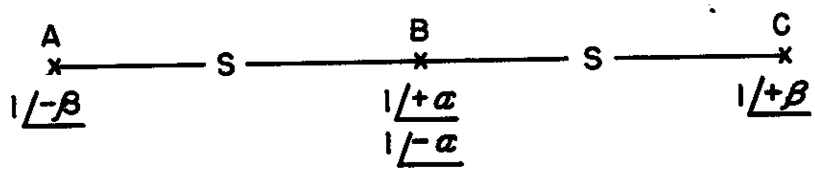
Any three element array regardless of the physical arrangements of the towers may be computed by using one element, or any arbitrary point in space as a reference point. Practical considerations will dictate that for equal height towers the reference point should be chosen symmetrically by space and time phase relation with regard to two elements. This permits evaluation in accordance with (12) above, and simplifies the computations. If one element is of different height, this element should be taken as the reference point for the computation of vertical sections. This applies also to a three-in-line configuration, because the simplification of the vertical section computations outweighs the possibly more cumbersome horizontal formula.

4. Four or more elements, all in line

Considerations leading to the development of the three-element array may be expanded to include additional elements up to any desired or practical limit. However, the method of development will depend on the desired configuration; especially on magnitude of null values, placement of towers of different height, and consideration of power distribution within the array. Various methematical short-cuts are sometimes possible which simplify the computations.

The analysis of equal-field components arrays shown by (1) to (4) and (5) to (6) above may be expanded to yield a four-element array as follows:

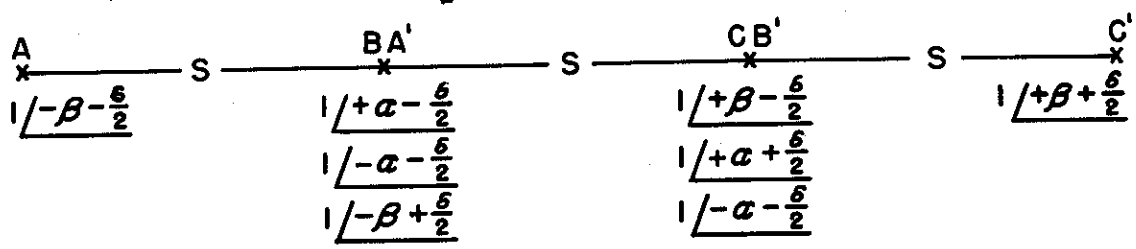
The configuration (4), constituting a resulting three-element array can be represented by



(13)

which is identical to (4) with $\beta = \frac{P}{2} + \frac{P'}{2}$ and $\alpha = \frac{P}{2} - \frac{P'}{2}$

Now assume a similar array A'B'C' placed such that elements A' and B, and B' and C coincide, at the same time offsetting elements A', B', C' by a phase angle δ from the corresponding elements A, B, C. A four-element array will result which, after subtracting $\frac{\delta}{2}$ from all phase angles looks as follows:



(14)

The vector equation for this array in its simplest form is based on the use of the midpoint between BA' and CB' as a reference point. It will assume the form

$$E = K' \left[m \cos\left(\frac{S}{2}\cos\Theta + \phi\right) + 2\cos\left(\frac{3S}{2}\cos\Theta + \psi\right)\right]$$

Where m, ϕ, and ψ are contractions of the vector combinations shown in (14). In general, actual numerical values will be easier to use than general formulas in every case where the number of elements exceeds three.

Similar operations will extend the number of elements to any desired value. The four-element array (14), for instance, is duplicated with the phase difference δ between corresponding elements. Now let three elements of each array coincide, and the five-element array is obtained. The vector-sum formula uses the center element as a reference point.

In general for multi-element arrays of this type computation is simplified if the phases are arranged symmetrically to a reference point which is the center element for arrays of an odd number of towers, and the midpoint of the array for an even number of towers.

A development analogous to (5) may be used to extend the multiplication formula to more than three elements. For unity field ratios in the components the half-angle formula may be used, and the general expression

$$E = K \cos\left(\frac{S}{2}\cos\Theta + \frac{P}{2}\right)\cos\left(\frac{S}{2}\cos\Theta + \frac{P'}{2}\right)\cos\left(\frac{S}{2}\cos\Theta + \frac{P''}{2}\right) \cdots\cdots$$

For an n element array the above expression will have $n-1$ terms. The numerical relation of the constant K for an n element array to the constant K' used for the vector-sum expression of the same array will be

$$K = 2^{n-1} K'$$

if unity field ratios exist in all components.

For different field ratios in the components the multiplication formula assumes the form

$$E = K \quad [R + \cos(S\cos\Theta + P)][R' + \cos(S\cos\Theta + P')][R'' + \cos(S\cos\Theta + P'')]\cdots\cdots$$

The vector equation for such an array may be developed by extending the procedure followed in (8), (9), and (10). A general expression would be rather cumbersome, but the use of specific parameters will permit simplifications. Care should be taken so that all phases are arranged symmetrically. The relation between the constants K for the multiplication formula for an n element array having the field ratios r, r', r'', and the constant K' for the vector sum formula is given as follows:

$$K' = \frac{K}{2^{\frac{n-1}{2}} \sqrt{r\,r'\,r''\cdots\cdots}}$$

where for each r:

$$R = \frac{1 + r^2}{2r}$$

The number of possible combinations which yield the identical horizontal plane pattern increases rapidly with the number of elements and may be visualized by setting up a table similar to (10a) to (10d). Considerations of vertical distribution (for unequal height towers), and current and power distribution, may influence the choice of the proper combination.

In analogy to (6) a quadrature component may be added to the center tower of any array consisting of an odd number of elements, which will provide equal field intensities in the nulls of the array. The added component "a" will appear as the quadrature term $\left(\frac{a}{2^{n-1}}\right)^2$ in the modified multiplication formula.

From a pure mathematical standpoint it is, of course, possible to add any component to any element of a multi-element array. However, the individual case will determine which procedure can be handled most conveniently from a practical view, considering the amount of computation work required, and the desired simplicity in derivation.

A four-element array with all elements in line and equally spaced may frequently be computed or analyzed in the following manner:

Two two-element components are selected which should have equal values of field intensity in their nulls. They are combined in accordance with (5), and a quadrature component "a" is added to the center-element of the three-element combination (6). This component three-element array is then repeated along the same axis so that there is a phase difference P'' and a field ratio r between corresponding elements. Then two elements of each array are made to coincide similar to (14) and the following equation results:

$$E = K\sqrt{\left[\cos^2\left(\frac{S}{2}\cos\Theta + \frac{P}{2}\right)\cos^2\left(\frac{S}{2}\cos\Theta + \frac{P'}{2}\right) + \frac{a^2}{4}\right]\left[R + \cos\left(S\cos\Theta + P''\right)\right]}$$

where

$$R = \frac{1+r^2}{2r}$$

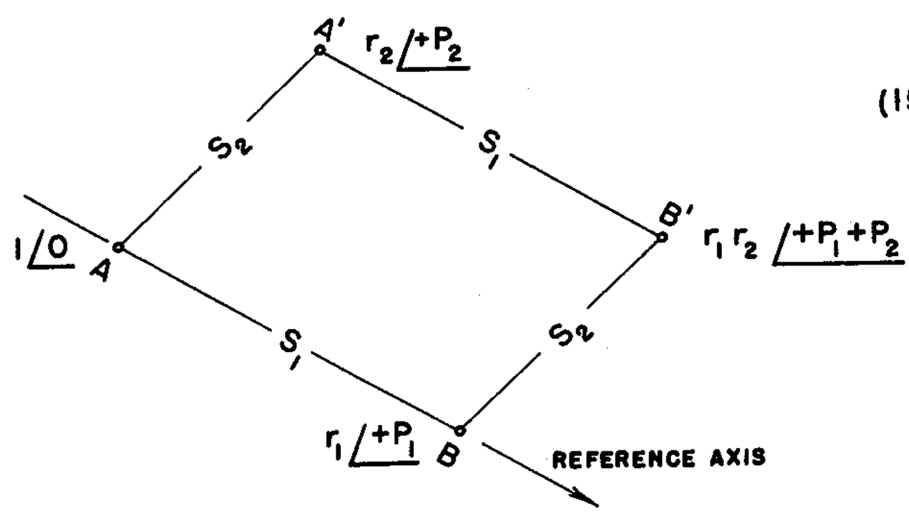
r is selected to give the desired null value for the third component. The vectors may be set up similar to (14), and the vector equation uses the midpoint of the array as a reference point. The constant K' for the vector equation is related to K as follows:

$$K' = \frac{K}{4\sqrt{2r}}$$

A general derivation of the vector equation again is cumbersome, and step-by-step computations using given parameters are much easier to handle.

5. Four or more elements, on corners of parallelogram.

For this configuration, an analysis is employed which is similar to the one used for certain three-element arrays:



(15)

The pair AB is spaced S_1 electrical degrees, a phase difference of P_1 degrees, and a field ratio r_1 . The reference axis is taken as zero degrees through the leading tower B, and its equation may be expressed as

$$\sqrt{R_1 + \cos(S_1 \cos\Theta + P_1)} \qquad \text{where } R_1 = \frac{1 + r_1^2}{2r_1}$$

As shown above, this field may be considered as originating from the midpoint between the two elements. Now take a second array A'B' oriented on a line parallel (but not necessarily collinear) to AB, which gives the same expression for its field if the relations between A' and B' are identical to A and B. If now A' and B' lead their corresponding elements A and B by P_2 degrees, radiate r_2 times the field radiated by A and B, and are each S_2 electrical degrees from their corresponding elements A and B, a two-element array is again formed by the midpoint of AB and A'B' which produce the field

$$\sqrt{R_2 + \cos(S_2 \cos\Theta' + P_2)}$$

where $R_2 = \dfrac{1 + r_2}{2r_2}$ and Θ' is counted from the reference axis joining the midpoints of the two arrays.

Combination of the two expressions is again by multiplication, and if the axis through AB is taken as a reference, the angle Θ counted from it differs by a degrees from the angle Θ' counted from AA'   BB'. The complete equation becomes

$$E = K\sqrt{\left[R_1 + \cos(S_1 \cos\Theta + P_1)\right]\left[R_2 + \cos(S_2 \cos[\Theta - a] + P_2)\right]}$$

where the angle Θ is counted from the axis AB toward that axis BB' which forms the angle a with AB.

If the arrays are collinear, then $a = 0$, and a four-in-line results which may be more useful in certain designs than the four-element arrays previously discussed.

For any particular case, a vector equation may be derived using any of the elements or any convenient point as a reference. The individual fields and phases for the elements appear in (15). Using these values in a vector equation, the constant K' of that equation is related to K above by the expression,

$$K' = \frac{K}{2\sqrt{r_1 \; r_2}}$$

which is identical to the relations for a three-element array described by (10).

The reasoning for the four-element parallelogram arrangement may be readily expanded to include more elements if they are arranged in parallel rows. The pattern produced by any particular row is considered first, and multiplied by the factor resulting from the combination of various rows. It is not practical to develop a general equation for an n element array of this kind.

The actual computation work may be considerably simplified by suitable trigonometric transformation if the given parameters permit such a procedure.

6. Other Configurations

Unsymmetrical configurations of four or more elements must be handled by ''brute force'' vector additions. This applies also to the vertical sections of any array having unequal tower height.

7. Computation of Mutual Impedance Values, Power Distribution Power Gain, etc.

A study of the vertical distribution for a single element shows that the ''effective field'' radiated along the horizontal plane varies with the electrical height. Without going into the physical basis and mathematical derivation, we can picture the three-dimensional radiation pattern as concentrating more or less energy at low vertical angles for varying height tower for the same amount of total energy radiated over the hemisphere. Numerically, the following formulas apply:

The vertical distribution for a single element is given as

$$f(V) = \frac{\cos H - \cos(H \sin V)}{(\cos H - 1)\cos V}$$

(The signs in numerator & denominator are wrong. The correct signs are as indicated)

where H is the electrical height and V is the vertical angle of elevation taken as zero at the groundplane.

The relation between base antenna current and unattenuated field at one mile is given by G. H. Brown as

$$E = 37.25\ I_0\ \frac{1 - \cos H}{\sin H}\ \text{MV/M}$$

where I_0 is in amperes.

For changes in antenna height both H and I_0 vary; the latter being dependent on radiation and loss resistance.

For a directional array we may obtain various types of vertical distribution which again may or may not concentrate energy along the horizontal plane for the same amount of total energy radiated over the hemisphere. If we compare the energy along the horizontal plane pattern for a specific directional array, and a single tower having the same height of all or the majority of towers in the array, we arrive at a definition of power or field gain. Obviously, this field gain is a geometric property of the array due to the shape of the three-dimension hemispherical pattern. However, it can be shown that these geometric relations may be computed and described in terms of current and impedance relations between the elements.

An analysis of this kind, therefore, permits a determination of the following quantities:

1. Operating impedances for each element.
2. Base currents for each element.
3. Power distribution within the array.
4. Power gain of the array measured along the horizontal plane and determination of actual values of radiated field produced by the array.

The analysis also permits allowance for loss resistance, or actual measured values of the resistance of individual elements.

Development of a complete equation covering all these points will not be attempted because it is not practical. A step-by-step consideration will permit determination of all desired quantities, and still preserve a clear picture of the problem and its engineering solution. The basis for this line of computation is the field and phase distribution of the various elements, and the self-resistance of each element. The fields and phases of the various towers are arranged so that one convenient element is taken as unity field at zero phase.

For theoretical consideration values of self-resistance for elements of various height have been computed; if desired, measured values may be used.

For each element the following relation exists: The voltage induced in each element is the vector sum of the voltages induced by the currents flowing in the element itself, and in all the other elements of the array.

$$V_1 = I_1 Z_1 = I_1 Z_{11} + I_2 Z_{21} + I_3 Z_{31} + \ldots\ldots$$

where I_1 is the operating current in tower 1, Z_1 its operating impedance, Z_{11} its self-impedance, $I_2, I_3 \ldots\ldots$ are currents flowing in elements 2, 3,...., and $Z_{21}, Z_{31} \ldots\ldots$ are the mutual impedance values between elements 2 and 1, 3 and 1, etc.

Dividing the expression by I_1, we obtain

$$Z_1 = Z_{11} + \frac{I_2}{I_1} Z_{21} + \frac{I_3}{I_1} Z_{31} \ldots\ldots$$

where all quantities are <u>vectors</u> and the actual computation is best performed using the polar form.

For equal height towers, the field ratio equals the current ratio. However, if the towers are not of equal height, the current ratio has to be determined by the use of the relation

$$E = 37.25\, I_0 \frac{1 - \cos H}{\sin H}$$ which gives for two towers

$$\frac{E_1}{E_2} = \frac{I_1}{I_2} \times \frac{1 - \cos H_1}{1 - \cos H_2} \times \frac{\sin H_2}{\sin H_1}$$

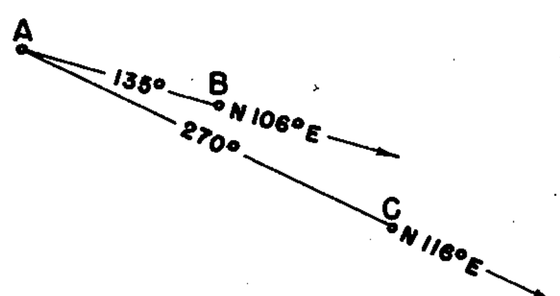
The values of mutual impedance for various assumptions of tower height and spacing may be computed using formulas developed by Russell Cox in I. R. E. Proceedings, November, 1947. For equal height towers not too different in electrical height from a quarter-wave sufficiently close results may be obtained by using values for quarter-wave towers which have been computed separately and are shown in graph form.

For all computations "per unit" values are easiest to handle, except when different tower heights are employed. In this case it may be more straightforward to use actual ohms in order to avoid errors in ratios.

After all values of mutual impedance are obtained, operating impedances and currents are determined for the desired power input. Comparison of the field intensity values obtained from the operating currents with the theoretical fields gives power gain or loss of the array.

The entire procedure is outlined below using an example with definite numerical values.

The directive array authorized for installation by WNDB, Daytona Beach, Florida, consists of three elements and will be operated on 1150 KC with 1 KW power during nighttime hours.



(16)

Pertinent constants and operating parameters for the towers are as follows:

Tower	Height	Field	Relative Phase
A	347'=146°	0.62	0°
B	220'=92.5°	0.61	-175°
C	220'=92.5°	1.00	+105°

The equation for this array uses the 146° high tower A as a reference point, and the line N 116° E as a reference axis, proceeding in a counter-clockwise direction. For an estimated R.M.S. value of 190 mv/m for 1 KW power, the equation is set up as follows:

$$E = 140.8\left\{0.62\, f_1(V)\underline{/0^\circ} + f_2(V)\left[0.61\underline{/135\cos(A-10)\cos V-175} + 1.0\underline{/270\cos A\cos V+105}\right]\right\}$$

where $f_1(V)$ and $f_2(V)$ are the vertical distribution factors for the 146° tower, and for the 92.5° towers, respectively.

For the first evaluation of operating impedances, power distribution, and power gain, all towers are considered thin radiators of the electrical height stated. As tower "A" is of different height, the current ratio is not equal to the field ratio. Retaining unity current in tower "C" we can write

$$\frac{E_A}{E_C} = \frac{I_A}{I_C} \times \frac{1-\cos H_A}{1-\cos H_C} \times \frac{\sin H_C}{\sin H_A}$$

which results in

$$I_A = \frac{E_A}{E_C} I_C \times \frac{1-\cos H_C}{1-\cos H_A} \times \frac{\sin H_A}{\sin H_C} = .198$$

The current ratios are, therefore, as follows:

Tower	Relative Current
A	.198
B	.61
C	1.0

The mutual impedance values between the towers are computed as stated before, and are found to be

$$Z_{AB} = Z_{BA} = 48.1 \angle -69.7^\circ \text{ ohms}$$

$$Z_{AC} = Z_{CA} = 29.4 \angle +162^\circ \text{ ohms}$$

$$Z_{BC} = Z_{CB} = 21.1 \angle -80^\circ \text{ ohms}$$

Z_{BC} was not computed in strict accordance with the theory, but as the height of towers B and C is very close to one-quarter wave, sufficiently close values are obtained by using the readily available mutual impedance values for 90° towers and correcting it by the ratio of the resistance values for 90° and 92.5° towers.

A graph of self-resistance versus electrical height for the radiators gives a value of 39 ohms for the 92.5° radiator, and 322.5 ohms for the 146° radiator.

Substituting in the expression

$$Z_1 = Z_{11} + \frac{I_2}{I_1} Z_{21} + \frac{I_3}{I_1} Z_{31} + \ldots\ldots$$

for the three elements in turn we first note that it is sufficient to take only the real (co-sine) or resistance component of the additive terms into account, because the reactance components do not affect the power distribution:

$$R_A = \mathcal{R}\left[322.5\angle 0^\circ + \frac{0.61\angle -175}{0.198\angle 0} \times 48.1\angle -69.7 + \frac{1.0\angle +105}{0.198\angle 0} \times 29.4\angle 162^\circ\right]$$

R_A = 251.4 ohms

$$R_B = \mathcal{R}\left[39\angle 0^\circ \quad + \frac{0.198\angle 0}{0.61\angle -175} \times 48.1\angle -69.7 + \frac{1.0\angle +105}{0.61\angle -175} \times 21.1\angle -80^\circ\right]$$

R_B = 2.4 ohms

$$R_C = \mathcal{R}\left[39\angle 0^\circ \quad + \frac{0.198\angle 0}{1.0\angle +105} \times 29.4\angle +162 \quad + \frac{0.61\angle -175}{1.0\angle +105} \times 21.1\angle -80^\circ\right]$$

R_C = 54.94 ohms

Knowing the operating resistance values, the power in the array per unit current in tower "C" equals the sum of the I^2R's in all towers. This may be set up more conveniently in form of a table as follows:

Tower	Operating Resistance	Unit Current	Unit Power
A	251.4	.198	9.86
B	2.4	.61	.89
C	54.94	1.0	54.94
		Total Unit Power.........	65.69

Proportional current for 1000 watts

input power.................. $1.0\sqrt{\frac{1000}{65.69}}$ = 3.90 amps.

We then set up:

Tower	Operating Resistance	Current for 1 KW input	Power Watts
A	251.4	0.772	150
B	2.4	2.38	13.6
C	54.94	3.90	836.

The power of all towers should add up to the total input power of 1000 watts (within the limits of the accuracy of the computations).

As the computations are based on unity current or field in tower "C", the field for the array can now be determined by use of the conversion formula

$$E_C = 37.25\, I_C \frac{1 - \cos H_C}{\sin H_C}$$

which gives 151.6. This, as apparent from the array formula where "C" is unity field, is the value of the array constant. This constant has the dimension of field intensity, and is directly proportional to the value of field intensity in any direction as well as to the R.M.S. value. As the assumed R.M.S. value of 190 mv/mresulted in a constant of 140.8, the theoretical R.M.S. value for the array, disregarding losses is computed as

$$\frac{151.6}{140.8} \times 190 = 204.8 \text{ mv/m}$$

This may be compared to the theoretical effective field of a single tower having the same height of towers B and C, or having the height of tower A. A relative value of field or power gain may be defined that way.

For consideration of losses it is usually sufficient to assume a value of loss resistance in series with each tower. Values range from 0.5 to 2.0 ohms; the latter being a fair average for medium soil, and a good ground system. For the WNDB array a two-ohm loss per tower would result in the following values:

Tower	Operating resistance including loss	Unit Current	Unit Power	Current for 1 KW
				amps.
A	253.4	.198	9.9	.757
B	4.4	.61	1.6	2.33
C	56.94	1.0	56.9	3.82
			68.4	

$$I_c = \sqrt{\frac{1000}{68.4}} = 3.82 \text{ amps.}$$

The 3.82 amps. value for tower "C" results in a constant 148.5, and an R.M.S. value of 200.5 mv/m per KW.

The analysis described above will serve to describe the array on a strictly theoretical basis. It would in itself indicate that the original assumption of a 190 mv/m R.M.S. was low, and should be increased to about 200 mv/m. It shows that theoretically this array could not develop an R.M.S. greater than 204.8 mv/m for 1 KW power input.

It is interesting to compare the obtained R.M.S. value with a value that could be arrived at under the assumption that all towers are of equal height (92.5°). In this case it is practical to use per unit values for mutual impedance which are as follows (computed for 90° towers, but close enough for the 92.5° elements):

$$Z_{AB} = .545 \angle -76^{\circ}$$
$$Z_{AC} = .345 \angle +157^{\circ}$$
$$Z_{BC} = .54 \angle -80^{\circ}$$

The self-resistance of the 92.5° element is 39.0 ohms. The current ratios may now be taken to equal the field ratios, and computations yield the following per-unit operating resistance values:

$$r_A = +0.748$$
$$r_B = +0.081$$
$$r_C = +1.461$$

For 39 ohm self-resistance the per unit values are multiplied by 39 to give the operating resistance values. The following table is set up:

Tower	Operating Resistance	Current Ratio	Unit Power	Current (amps)
A	29.2	0.62	11.2	2.35
B	3.16	0.61	1.2	2.31
C	57.0	1.0	57.0	3.79
			69.4	

$$I_C = \sqrt{\frac{1000}{69.4}} = 3.79$$

The constant is again defined as

$$K = 37.25\, I_C \frac{1 - \cos H}{\sin H} = 147.8$$

$$\text{R.M.S.} = \frac{147.8}{140.8} \times 190 = 199.5 \text{ mv/m}$$

This may be compared to the value obtained by using the actual tower height (204.8 mv/m). The field gain due to one higher tower, in the case of this particular array, amounts to 2.7% which is apparent from the ratio of the R.M.S. value. The reason for a horizontal field gain is that the one higher tower in this particular array does suppress vertical radiation and tends to concentrate the field at low angles near the horizontal plane.

8. Use of Measured Data and Integration of Such Data into Computations of Effective Field and Power Distribution

The use of measured resistance data would, off-hand, not appear to present particular problems. We may, for the simplest case, split a measured base resistance R_M into a value R_0 and a 2 ohm loss resistance so that $R_0 = R_M - 2$
The R_0 part is used as a basis for mutual impedance values, and the 2 ohm loss resistance is added to the operating resistance values obtained.

This method, straight-forward as it may seem, may lead to amazingly impossible results. The widest variations will be apparent (a) if the tower height approaches one-half wave, and (b) if the tower is self-supporting and has a relatively broad base. The theoretical resistance of a half-wave radiator approaches infinity. In practice, values of several hundred ohms have been measured. Allowance should also be made for the velocity of propagation along the radiator which makes the tower appear to be electrically higher than its physical length (the increase usually being taken as 5%).

Of course, if mutual impedance values between dissimilar towers (in height, shape, or both) are computed, values found by the theoretical assumption of infinitely thin radiators cannot be directly applied if actual base resistance measurements show a discrepancy from the theoretical value.

For example, in the WNDB array described above, the high tower A is existing, and its base resistance has been measured. It is a self-supporting, tapered, triangular tower. The resistance, as measured, is 221 ohms, which is considerably lower than the theoretical 322.5 ohm value, and the base current will be 2.13 amps. per kilowatt. Assuming the theoretical current distribution of a 146^{o} high tower, and 1 KW power input, this tower would develop an effective field of 259 mv/m for a base current of 2.13 amps., which represents an increase of 21% over the theoretical effective field of 214 mv/m for a 146^{o} tower. An increase of this kind has not been observed in actual practice. If the apparent height increase due to the velocity of propagation is considered, the discrepancy becomes even more glaring.

In general, self-supporting towers have lower base resistance values than uniform cross-section guyed towers, and these values are considerably below the theoretical values for tower heights approaching one-half wave.

Of course, such discrepancies would tend to invalidate the results of mutual impedance calculations giving unreliable or even impossible values. In the instant example, we should not use a base resistance value of 221 ohms in conjunction with mutual impedance values based on 322.5 ohms for the high tower, especially as there is reasonably close agreement of theoretical and measured values for the short towers which are uniform cross-section.

We may, of course, assume that the current distribution in a tapered tower is not the same as in an infinitely thin radiator, and may not be even sinusoidal. The exact nature of that distribution seems to be beyond the scope of the mathematics usually associated with the theory.

In the case of towers between 120 and 160 electrical degree height, the theoretical resistance varies rapidly for even small height changes. It may be suggested that the measured base resistance is taken as the determining factor, and the tower is assumed to have a sinusoidal current distribution for the height corresponding to the base resistance for a thin tower having sinusoidal distribution. This approach should be used for the purpose of determining mutual impedance values only.

This method has been tried in the WNDB case using the following procedure:

(a) The measured 221 base resistance of the existing tower "A" was assumed to consist of a 219 ohm radiation term, and a 2 ohm loss resistance. The 219 ohm radiation resistance corresponds to a 139° high "ideal" (infinitely thin) radiator.

(b) The new towers are expected (from data supplied by manufacturers) to have a base resistance of 45 ohms. This is taken as the sum of a 43 ohm radiation term, and a 2 ohm loss term. The 43 ohm radiation resistance corresponds to a 95° high "ideal" tower.

(c) Mutual impedance values are now computed for the "equivalent" tower heights obtained. These values are:

$Z_{AB} = 44.9 \angle -71.6^{\circ}$

$Z_{AC} = 28.1 \angle +161^{\circ}$

$Z_{BC} = 23.4 \angle -80^{\circ}$ (This Z_{BC} value taken proportional to a 90° mutual impedance.)

(d) The current ratio between towers was now based on the comparison of 95° and 139° towers. ($I_A : I_B : I_C = 0.253 : 0.61 : 1.0$

(e) Based on (c) and (d) the following base operating resistance values and base currents were obtained (after adding 2 ohms loss resistance to each tower:)

R_A = 169.2 ohms I_A = 0.918 amps

R_B = 4.6 ohms I_B = 2.21 amps

R_C = 63.25 ohms I_C = 3.63 amps

Based on the value for I_C , and the equivalent tower height for "C", the constant K is found to be 147.5, resulting in an R.M.S. value of 199 mv/m.

It was not necessary to take velocity of propagation into account. The analysis is based on an assumed height and current distribution anyway, and the only physical fact considered is the measured base resistance of the employed tower, or a similar radiator.

All these various methods show that, in this particular case, at least, there is not too much variation in the result. However, the data computed provides a range of operating values which may reasonably be expected from the array. Components of phasing gear should be designed to take care of at least those variations which are indicated by the different methods of computations.

The results also indicate that the R.M.S. estimated for the WNDB directive array was low, and the design should be modified for an R.M.S. value of about 200 mv/m instead of 190 mv/m. However, specified tolerances (or M.E.O.V.) will permit the increase without violation of any protection requirements.

In case of towers of unequal height, it may be possible to compute operating data on the basis of loop current and loop resistance values rather than base current and base resistance values. Such an approach should be based on the primary integration formulas appearing in the pertinent literature.